MATHEMATICS

APPROXIMATION BY LACUNARY POLYNOMIALS *

BY

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INTRODUCTION

In this paper we will be concerned with certain refinements of classical complex approximation theorems of Runge and Walsh. The results to be discussed might be called lacunary versions of the classical theorems, that is, we will be concerned with uniform approximation of a holomorphic function by lacunary polynomials. Here the word lacunary is used loosely; it will mean, quite simply, that not all powers of z will appear in the approximating polynomials.

The result of Walsh referred to above states [12, 13]: Let D be the region interior to a simple closed curve γ in the plane. If $f(z)$ is holomorphic on D and continuous on $\bar{D}$, there exists a sequence of polynomials which converges uniformly to $f(z)$ on $\bar{D}$. The simplest form of Runge's theorem [11] asserts that if D is a simply connected open set in the plane, and if $f(z)$ is holomorphic on D, there exists a sequence of polynomials which converges uniformly to $f(z)$ on every compact subset of D.

We are interested in being able to approximate the function $f(z)$ uniformly on $\bar{D}$ (or on the compact subsets of D) by polynomials $P(z)$ of the form $\Sigma b_n z^{p_n}$, where $0 \leqslant p_0 < p_1 < \ldots$ is a fixed subsequence of the nonnegative integers. More specifically, we ask under what conditions on the domain D, the function $f(z)$ and the sequence of exponents $\{p_n\}$ can the approximating polynomials be chosen from such a lacunary subclass? Consideration of this class of polynomials implies a special role for the origin; accordingly, it is assumed throughout the paper that 0 belongs to D. It is natural then, that all results contain the hypothesis that $f(z)$ have an expansion of the form $f(z) = \sum_{n=0}^{\infty} a_n z^{p_n}$ near $z=0$. No other functions could be uniform limits, on a neighborhood of 0, of polynomials which contain only powers z^{p_n}.

§ 1. SUMMARY OF RESULTS

We indicate only the results of Walsh type – that is, results dealing

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with uniform approximation on the closure of a Jordan domain. The results of Runge type (Theorems 1, 3 and 5) are similar.

Let $f(z)$ be holomorphic on a Jordan domain D and continuous on $\bar{D}$. Assume that $0 \in D$ and that

$$f(z) = \sum_{n=0}^{\infty} a_n z^{p_n} \tag{1}$$

near $z=0$, where $0 \leqslant p_0 < p_1 < p_2 \ldots$ is a subsequence of the nonnegative integers. Under each of the following conditions we are able to assert that $f(z)$ can be uniformly approximated on $\bar{D}$ by polynomials of the form $\Sigma b_k z^{p_k}$.

THEOREM 2. If D is strongly starlike relative to $z=0$, that is, each ray from the origin intersects ∂D exactly once (p_n arbitrary).

THEOREM 4. If $\gamma = \partial D$ is of class C^1 (that is, γ has continuously turning tangent line) and $\{p_n\}$ has density 0 ($n/p_n \to 0$).

THEOREM 6. If $\gamma = \partial D$ is of class C^1 and $\{p_n\}$ has density 1 ($n/p_n \to 1$).

In addition, examples are given which show that for each $\delta \in (0, 1)$ there is a sequence $\{p_n\}$ having density δ, a Jordan domain D containing the origin which is arbitrarily close to strongly starlike, and a function $f(z)$ holomorphic on D, continuous on $\bar{D}$ and having an expansion (1) near $z=0$, such that $f(z)$ can not be approximated uniformly on $\bar{D}$ by polynomials of the form $\Sigma b_k z^{p_k}$.

The main tools in our arguments are the Mittag-Leffler summability method (Theorem A), the Fabry-Pólya gap theorem (Theorem B), and a local version of the F. and M. Riesz Theorem (Theorem C).

§ 2. MITTAG-LEFFLER SUMMABILITY AND THE FABRY-PÓLYA THEOREM

Certain known results which will be used in our proofs are stated here along with some relevant definitions. A domain D (or more general region) in the plane is said to be starlike relative to the origin if whenever $z \in D$, the closed segment from 0 to z also lies in D. For brevity, such domains will sometimes be called starlike. With any function $f(z)$, holomorphic in a neighborhood of the origin, there is associated an open starlike subset of the plane called the Mittag-Leffler star of $f(z)$. To construct this set, rays from the origin are drawn through each singular point of $f(z)$, and the closed part of each ray from the singular point out to infinity is deleted. For example, the Mittag-Leffler star for $1/(1-z^2)$ is the set $C \backslash \{[1, +\infty) \cup (-\infty, -1]\}$.

A series $\sum_{n=0}^{\infty} \alpha_n$ of complex numbers is said to be Mittag-Leffler summable to the value A if

$$\sum_{n=0}^{\infty} \frac{\alpha_n}{\Gamma(1+n\delta)} \text{ is convergent for every } \delta > 0,$$

and

$$\lim_{\delta\to 0+} \sum_{n=0}^{\infty} \frac{\alpha_n}{\Gamma(1+n\delta)} = A.$$

We will use the following result in section 3.

THEOREM A. ([8]; cf. [4], ch. VIII). Let $f(z)$ be holomorphic in a neighborhood of the origin and have the expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2}$$

near $z=0$. Then

$$\lim_{\delta\to 0+} \sum_{n=0}^{\infty} \frac{a_n z^n}{\Gamma(1+n\delta)} = f(z),$$

uniformly on every compact subset of the Mittag-Leffler star for $f(z)$.

This method provides a way to sum the series $\Sigma a_n z^n$ to $f(z)$ throughout the largest starlike domain on which $f(z)$ is holomorphic. Such a set may well be considerably larger than the disc where the expansion (2) is valid.

For an increasing sequence $0 \leqslant p_0 < p_1 < \ldots$ of nonnegative integers, and any number $t > 0$, let $N(t)$ be the number of integers n for which $p_n \leqslant t$. We say that the sequence $\{p_n\}$ has density ϱ if $\lim_{t\to\infty} N(t)/t = \varrho$. This limit need not exist in general, but if it does, its value is the same as that of $\lim_{n\to\infty} n/p_n$ (it is assumed here that the sequence $\{p_n\}$ is infinite). If $\delta\{p_n\}$ denotes the value of either limit, it is clear that $0 \leqslant \delta\{p_n\} \leqslant 1$, and that $\delta\{p_n\} = 0$ if and only if $p_n/n \to \infty$. Denoting by $q_0 < q_1 < q_2 < \ldots$ the sequence of nonnegative integers complementary to $\{p_n\}$, $\delta\{p_n\}$ will exist if and only if $\delta\{q_n\}$ exists, and $\delta\{p_n\} + \delta\{q_n\} = 1$. Thus $\delta\{p_n\} = 1$ if and only if $q_n/n \to \infty$ (assuming $\{q_n\}$ is infinite).

The following theorem is essentially due to Fabry ([2], [3], cf. [1]), and is contained in a generalization due to Pólya ([9], p. 626). In this and all other results, $\{p_n\}$ will be a sequence of integers satisfying $0 \leqslant p_0 < p_1 < p_2 < \ldots$.

THEOREM B. Let $f(z)$ be holomorphic at $z=0$ with

$$f(z) = \sum_{n=0}^{\infty} a_n z^{p_n}, \quad |z| < R,$$

where R is the radius of convergence of the series.

(i) If $n/p_n \to 0$, every point of the circle $\{|z| = R\}$ is a singular point of $f(z)$.

(ii) If $n/p_n \to \delta$, $0 < \delta < 1$, every closed arc of $\{|z| = R\}$ of length $2\pi R\delta$ contains a singular point of $f(z)$.

§ 3. STARLIKE DOMAINS

The proof of Walsh's theorem is particularly simple for the case of a domain D which is strongly starlike relative to $z=0$ (cf. [13], p. 36). The standard proofs do not seem to extend to the lacunary case, but one can use Mittag-Leffler summability to obtain the desired results. We first consider the simpler Runge-type question.

THEOREM 1. Let D be an open set, starlike relative to $z=0$, and let $f(z)$ be holomorphic on D with

$$f(z)=\sum_{n=0}^{\infty} a_n z^{p_n} \tag{3}$$

near $z=0$. Then there exists a sequence of polynomials $P_n(z)$, containing only powers z^{p_k}, such that

$$\lim_{n\to\infty} P_n(z)=f(z),$$

uniformly on every compact subset of D.

PROOF. Let $K_1 \subset K_2 \subset \ldots \subset D$ be a sequence of compact subsets of D such that $\bigcup_{n=1}^{\infty} K_n = D$. We require also that for any compact $K \subset D$, there exists an integer $n_0 = n_0(K)$ such that $K \subset K_n$ for all $n \geqslant n_0$. It suffices to exhibit for each n a polynomial $P_n(z)$ containing only powers z^{p_k} for which $\|f-P_n\|_{K_n} < 1/n$. The norm used throughout is the supremum norm over the indicated set.

Since D is a subset of the Mittag-Leffler star for $f(z)$, Theorem A assures for each n the existence of a number $\delta_n > 0$ such that

$$\Big\| \sum_{k=0}^{\infty} \frac{1}{\Gamma(1+\delta_n p_k)} a_k z^{p_k} - f(z) \Big\|_{K_n} < \frac{1}{2n}.$$

Thus for any N,

$$\Big\| \sum_{k=0}^{N} \frac{1}{\Gamma(1+\delta_n p_k)} a_k z^{p_k} - f(z)\Big\|_{K_n} \leqslant \frac{1}{2n} + \Big\| \sum_{k=N+1}^{\infty} \frac{1}{\Gamma(1+\delta_n p_k)} a_k z^{p_k} \Big\|_{K_n}$$
$$\leqslant \frac{1}{2n} + \sum_{k=N+1}^{\infty} \frac{|a_k|}{\Gamma(1+\delta_n p_k)} \varrho^{p_k},$$

where $\varrho = \max_{z\in K_n} |z|$. Since (3) has a nonzero radius of convergence, the series

$$\sum_{k=0}^{\infty} \frac{a_k z^{p_k}}{\Gamma(1+\delta_n p_k)}$$

converges for all z, and hence absolutely for $z=\varrho$. We thus can choose N sufficiently large so that

$$\| \sum_{k=0}^{N} \frac{1}{\Gamma(1+\delta_n p_k)} a_k z^{p_k} - f(z) \|_{K_n} < \frac{1}{n}.$$

The corresponding lacunary version of Walsh's theorem requires that the Jordan domain D be strongly starlike relative to $z=0$, that is, that each ray from the origin intersects ∂D exactly once. This property guarantees that for every number $\varrho > 1$, the domain $\varrho D = \{\varrho z | z \in D\}$ contains the set $\bar{D}$.

THEOREM 2. Let D be a Jordan domain, strongly starlike relative to $z=0$, and let $f(z)$ be holomorphic on D, continuous on $\bar{D}$, with

$$f(z) = \sum_{n=0}^{\infty} a_n z^{p_n}$$

near $z=0$. Then there exists a sequence of polynomials $P_n(z)$ containing only powers z^{p_k} such that

$$\lim_{n\to\infty} P_n(z) = f(z),$$

uniformly on $\bar{D}$.

PROOF. Let $\varepsilon > 0$ be given. Since $f(z)$ is continuous on the compact set $\bar{D}$, we can choose $\delta > 0$ so that $|f(z)-f(z')| < \varepsilon/2$ whenever $z, z' \in \bar{D}$ and $|z-z'| < \delta$. We next determine a number θ, $0<\theta<1$, so close to 1 that $|z-\theta z| < \delta$ for all $z \in \bar{D}$. Since D is starlike, θz belongs to $\bar{D}$ whenever $z \in \bar{D}$. We thus have $\|f(z) - g_\theta(z)\|_{\bar{D}} < \varepsilon/2$, where $g_\theta(z) = f(\theta z)$.

The function $g_\theta(z)$ is holomorphic on the starlike domain $(1/\theta)D$ which contains $\bar{D}$ as a compact subset. Since $g_\theta(z)$ has the lacunary expansion

$$g_\theta(z) = \sum_{n=0}^{\infty} a_n \theta^{p_n} z^{p_n}$$

near $z=0$, Theorem 1 guarantees the existence of a polynomial $P(z)$ of the desired type for which $\|P(z) - g_\theta(z)\|_{\bar{D}} < \varepsilon/2$, and hence $\|P(z) - f(z)\|_{\bar{D}} < \varepsilon$.

It should be emphasized that no condition on the sequence $\{p_n\}$ is required in Theorems 1 and 2.

§ 4. NON-STARLIKE DOMAINS: COUNTEREXAMPLES

When the domain D fails to be starlike, lacunary versions of the Runge and Walsh theorems cannot be true without strong restrictions on the sequence of exponents $\{p_n\}$. For very regular sequences $\{p_n\}$, the value δ of the density plays a crucial role. Counterexamples are given here for the Walsh question. Minor modifications would provide examples demonstrating the impossibility of Runge-type approximation in the case of general D and $\{p_n\}$.

First example: $\delta=1/k$. Let $f(z)=1/(1-z^k)$, where k is an arbitrary positive integer. For $|z|<1$ we have

$$f(z)=\sum_{n=0}^{\infty} z^{kn}\ : \tag{4}$$

the exponents $p_n=kn$ have density $1/k$. Let D be the Jordan domain indicated in figure 1A.

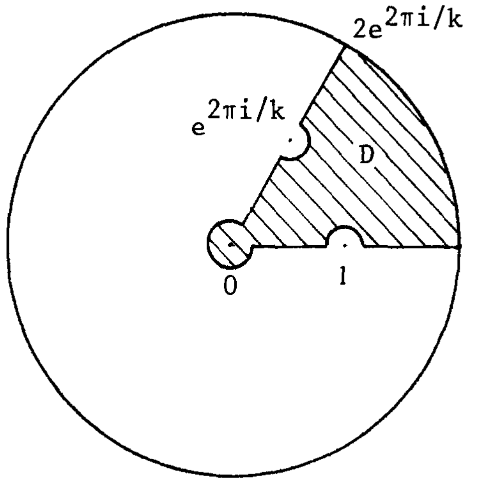

Fig. 1A.

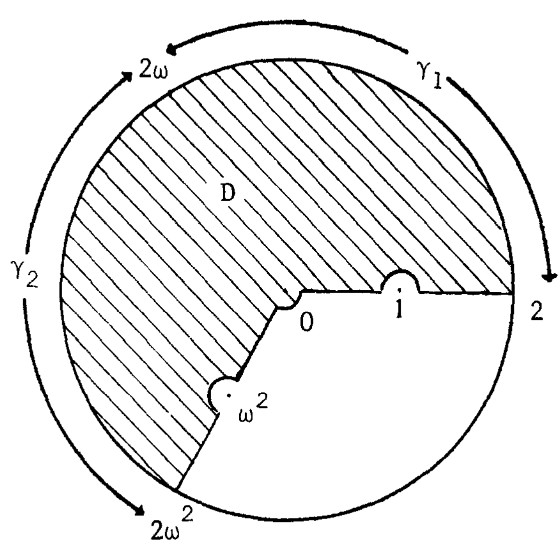

Fig. 1B.

Although $f(z)$ is holomorphic on D, continuous on $\bar{D}$, and has the expansion (4) around $z=0$, it is not possible to approximate $f(z)$ uniformly on $\bar{D}$ by polynomials $P_\nu(z)$ which contain only powers z^{kn}. Indeed, if there were such polynomials we would have $\|P_\nu - f\|_\Gamma \to 0$, where Γ is the arc $\{z=2e^{it}|0 \leqslant t \leqslant 2\pi/k\}$. But $P_\nu(z)-f(z)$ is a function of z^k and hence its values for $z \in \Gamma$ are reproduced on each of the arcs $e^{2\pi i j/k}\Gamma$, $j=1,2,\ldots,k-1$. Thus uniform convergence of P_ν to f on Γ would imply that the P_ν converge uniformly on $\{|z|=2\}$, and hence for $|z| \leqslant 2$: as a consequence $f(z)$ would have a holomorphic extension to the disc $\{|z|<2\}$! Observe that the boundary of D could be modified so that it is of class C^∞ and arbitrarily close to starlike.

The general case of rational density between 0 *and* 1. A similar counter-example can be constructed for any rational density $\delta = h/k$, $0<h/k<1$. We begin by carrying out the work for density 2/3.

Second example: $\delta = 2/3$. Let $\omega = e^{2\pi i/3}$ and

$$f(z) = \frac{z(z-\omega)}{z^3-1},$$

so that $f(z)$ has the lacunary expansion

$$f(z) = \omega z - z^2 + \omega z^4 - z^5 + \ldots$$

around $z=0$. The function $f(z)$ is holomorphic on the domain D indicated in fig. 1B and continuous on $\bar{D}$. If there were a sequence of polynomials $P_\nu(z)$, containing only powers occurring in the series for $f(z)$, for which $\|P_\nu - f\|_{\bar{D}} \to 0$, then in particular $\|P_\nu - f\|_{\gamma_1 \cup \gamma_2} \to 0$. We will show below that uniform convergence of $\{P_\nu\}$ on $\gamma_1 \cup \gamma_2$ implies uniform convergence on the circle $\{|z|=2\}$. But this would imply that $f(z)$ has a holomorphic extension to $\{|z|<2\}$, contradicting the existence of poles at ω^2 and $\omega^3=1$.

To complete the argument, suppose there were polynomials of the form

$$P_\nu(z) = zq_\nu(z^3) + z^2 r_\nu(z^3),$$

where q_ν and r_ν are polynomials, with $\|P_\nu - f\|_{\bar{D}} \to 0$. It is easy to see that the sequences $q_\nu(z^3)$ and $r_\nu(z^3)$ must converge uniformly on γ_1, and hence also on $\{|z|=2\}$. Indeed, if $z \in \gamma_1$ so that $\omega z \in \gamma_2$, then by assumption both

$$P_\nu(z) = zq_\nu(z^3) + z^2 r_\nu(z^3)$$

and

$$P_\nu(\omega z) = \omega z q_\nu(z^3) + \omega^2 z^2 r_\nu(z^3)$$

converge uniformly on γ_1. Combination shows the uniform convergence of $q_\nu(z^3)$ and $r_\nu(z^3)$ on γ_1, and the periodicity of the latter guarantees their uniform convergence on $\{|z|=2\}$.

For general rational density $h/k \in (0, 1)$ we use the function

$$f(z) = \frac{z(z-\omega) \ldots (z-\omega^{h-1})}{z^k-1},$$

where now $\omega = e^{2\pi i/k}$. D is constructed very much as before: it is approximately a circular sector of radius 2 and angular opening $2\pi h/k$ at the origin, it contains the points $\omega, \ldots, \omega^{h-1}$ but no other kth roots of unity, and its boundary contains the arc Γ: $\{z=2e^{it}|0 \leqslant t \leqslant 2\pi h/k\}$. The density δ of the exponents $\{p_n\}$ associated with the nonzero terms in the expansion of $f(z)$ around $z=0$ is equal to h/k. Indeed, by inspection $\delta \leqslant h/k$. By the Fabry-Pólya theorem, every arc of $\{|z|=1\}$ of length $> 2\pi\delta$ must contain a singular point of $f(z)$. But $f(z)$ is regular on the arc $\{z=e^{it}|0<t<2\pi h/k\}$, hence $\delta \geqslant h/k$.

As in the case where $\delta = 2/3$, one can show that a sequence of polynomials $P_\nu(z)$ (containing only powers of z occurring in the expansion of $f(z)$) which converges uniformly on Γ, also converges uniformly on $\{|z|=2\}$. However, since the latter is not possible if $P_\nu(z) \to f(z)$ on $\bar{D}$, $f(z)$ cannot be the uniform limit on $\bar{D}$ of such a sequence $P_\nu(z)$. The proof makes use of the nonvanishing of the Vandermonde determinant associated with $z_1=\omega,\ z_2=\omega^2,\ \ldots,\ z_h=\omega^h$.

The case of irrational density. We finally indicate how one can obtain a similar counterexample for the case of irrational density δ. By the theory of approximation with continued fractions, δ has infinitely many representations

$$\delta = h/k + \varepsilon \text{ with } 0<\varepsilon<1/k^2$$

(cf. [5], p. 140; we will only use the fact that $0<\varepsilon<1/2k$). For such a representation, we consider a function of the form

$$f(z) = f_1(z) + f_2(z),$$

where $f_1(z)$ is the function used in the case of density h/k above. For $f_2(z)$ we take a power series with regular exponents μ_n whose density is ε. We take the μ_n different from the exponents in $f_1(z)$; multiples of k will do, for example,

$$\mu_n = k\left[\frac{n}{k\varepsilon}\right],$$

so that

$$\mu_{n+1} - \mu_n > \frac{1}{\varepsilon} - k > \frac{1}{2\varepsilon},\ n=1,\ 2,\ \ldots.$$

The coefficients in $f_2(z)$ are chosen so that the power series has radius of convergence >2. For D we take a domain obtained from that for the case of density h/k by adding the intersection of $\{|z|<2\}$ with a small disc about the point $2\exp(2\pi ih/k)$. The radius must be such that the boundary ∂D contains the arc

$$\Gamma\colon \left\{ z = 2e^{it} | 0 \leqslant t \leqslant 2\pi(\delta+\eta) = 2\pi\left(\frac{h}{k}+\varepsilon+\eta\right)\right\}.$$

Here $\eta>0$ is to be chosen such that $f(z)$ cannot be a uniform limit of polynomials $P_\nu(z)$ on $\bar{D}$, where the $P_\nu(z)$ contain only powers occurring in $f(z)$. (In our case, any $\eta>\varepsilon$ will do).

For the verification we will need the following theorem of A. E. Ingham [7] (cf. Hayman [6]). For every $\varrho>1$ there exists a constant $B(\varrho)$ so that for all exponential sums

$$g(t) = \Sigma c_n e^{i\mu_n t}$$

with μ_n real and $\min(\mu_{n+1}-\mu_n) \geqslant A > 0$,

$$\int_0^{2\pi} |g(t)|^2 dt \leqslant AB(\varrho) \int_0^{2\pi\varrho/A} |g(t)|^2 dt.$$

We will give the details for the case $h/k=2/3$.

Third example: $\delta = 2/3+\varepsilon$, $0<\varepsilon<1/6$. Let

$$f(z) = \frac{z(z-\omega)}{z^3-1} + \sum_1^\infty \left(\frac{z}{3}\right)^{\mu_n}, \quad \omega = e^{2\pi i/3}, \quad \mu_n = 3\left[\frac{n}{3\varepsilon}\right],$$

$$P_\nu(z) = zq_\nu(z^3) + z^2 r_\nu(z^3) + s_\nu(z^3),$$

where q_ν, r_ν, s_ν are polynomials, $s_\nu(z^3) = \Sigma b_{\nu n} z^{\mu_n}$. We suppose that $P_\nu \to f$ uniformly on $\bar{D}$, and in particular on the arc $\Gamma\colon \{z=2e^{it} | 0 \leqslant t \leqslant 2\pi(2/3+\varepsilon+\eta)\}$. Then the polynomials

$$P_\nu(z) + P_\nu(\omega z) + P_\nu(\omega^2 z) = 3s_\nu(z^3)$$

will converge uniformly for $z=2e^{it}$, $0 \leqslant t \leqslant 2\pi(\varepsilon+\eta)$. With $\eta>\varepsilon$, we may apply Ingham's result to $g(t) = s_\alpha(2^3 e^{3it}) - s_\beta(2^3 e^{3it})$, taking $A=1/2\varepsilon$, $\varrho = A(\varepsilon+\eta)$. The conclusion is that the polynomials $s_\nu(z^3)$ are L^2 convergent on the circle $\{|z|=2\}$, hence the polynomials $zq_\nu(z^3)+z^2r_\nu(z^3)$ will be L^2 convergent on Γ. The method of example 2 now shows that the latter are L^2 convergent on $\{|z|=2\}$, hence the P_ν are L^2 convergent on that circle, and therefore (by the Cauchy representation) uniformly convergent for $|z| \leqslant 3/2$ —— contradicting the form of $f(z)$.

The above examples show that for every number δ between 0 and 1, there is a sequence of integers $\{p_n\}$ of density δ, a domain D containing the origin which is very close to starlike, and a function $f(z)$ continuous on $\bar{D}$ and holomorphic on D which has the local representation $\Sigma a_n z^{p_n}$, but which is not uniformly approximable on $\bar{D}$ by polynomials involving only powers z^{p_n}.

§ 5. EXPONENTS OF DENSITY ZERO

The examples of the previous section leave open the possibility that for exponents of density zero or one, we can approximate $f(z)$ by lacunary polynomials on non-starlike domains. Using the Fabry gap theorem we will prove in this section that such is the case when $\delta = 0$. Similar results for $\delta = 1$ are more difficult to obtain and are contained in sections 6–8.

THEOREM 3. Let D be an open connected set in the complex plane with $0 \in D$. Let $f(z)$ be holomorphic on D with

$$f(z) = \sum_{n=0}^{\infty} a_n z^{p_n} \tag{5}$$

near $z = 0$, where $\{p_n\}$ has density 0. Then there exists a sequence of polynomials $\{P_k(z)\}$, containing only powers z^{p_n}, such that $\lim_{k\to\infty} P_k(z) = f(z)$, uniformly on every compact subset of D.

PROOF. Let $\Delta_\varrho = \{|z| < \varrho\}$ be the smallest open disc centered at 0 which contains D (ϱ may be infinite). Since D is connected, every circle $T_r = \{|z| = r\}$, $r < \varrho$, contains a point of holomorphy of $f(z)$. By the Fabry gap theorem, the radius of convergence R of (5) must be $\geqslant \varrho$. But then (5) converges uniformly on every compact subset of D, so that for $P_k(z)$ we may use the partial sums of the series.

In order to prove a lacunary Walsh result for exponents of density 0, it is convenient to assume that the boundary of D satisfies a weak smoothness condition. Specifically, let D be the Jordan domain interior to a simple closed curve γ, and let $\Delta_\varrho = \{|z| < \varrho\}$ be the smallest open disc centered at the origin which contains D. By an extreme point of γ we mean any point $\zeta \in \gamma \cap \partial\Delta_\varrho$. For the next result we require that γ be smooth in a neighborhood of each of its extreme points.

THEOREM 4. Let D be a Jordan domain with $0 \in D$. Let $f(z)$ be holomorphic on D, continuous on $\bar{D}$, with

$$f(z) = \sum_{n=0}^{\infty} a_n z^{p_n}$$

near $z = 0$. Suppose that $\gamma = \partial D$ is of class C^1 in a neighborhood of each of its extreme points (or at least, that D coincides locally with a starlike domain there), and that $\{p_n\}$ has density zero. Then there exists a sequence of polynomials $\{P_k(z)\}$, containing only powers z^{p_n}, for which $\lim_{k\to\infty} P_k(z) = f(z)$, uniformly on $\bar{D}$.

PROOF. Let K be the smallest compact, starlike set which satisfies $\bar{D} \subset K \subset \bar{\Delta}_\varrho$. An essential property of K following directly from the assumption on γ is:

(6) For each extreme point $\zeta \in \gamma$, there exists a neighborhood $N_\delta(\zeta) = \{|z-\zeta| < \delta\}$ for which $N_\delta(\zeta) \cap \bar{D} = N_\delta(\zeta) \cap K$.

Thus no points of $K \backslash \bar{D}$ occur near an extreme point of γ.

The Fabry gap theorem guarantees that the series for $f(z)$ must converge for $|z| < \varrho$, so that f has a holomorphic extension to Δ_ϱ. Using (6), we can show that (the extended) $f(z)$ is continuous on K. Indeed, any point of K which is not an extreme point of γ lies in Δ_ϱ and hence is a point of continuity of $f(z)$. For a point $\zeta \in K$ which is an extreme point of γ, $f(z)$ is assumed continuous at ζ relative to $\bar{D}$, and hence by (6) is also continuous at ζ relative to K. The desired result now follows from the proof of Theorem 2, with K in the role of $\bar{D}$; the function $f(\theta z)$ will be holomorphic for $|z| < \varrho/\theta$.

§ 6. RUNGE-TYPE RESULT FOR DENSITY ONE

There is one particularly simple case of an exponent sequence $\{p_n\}$ of density one where our results are easy to obtain. This occurs when the set $\{q_n\}$ of nonnegative integers complementary to $\{p_n\}$ is finite in number. In this case the approximation theorems of both Runge and Walsh type are true without any special geometric restriction on D or smoothness condition on ∂D. Indeed, uniform convergence of polynomials to a function $f(z)$ of the form (1), on a neighborhood of 0, implies that the coefficients of the powers z^{q_n} in the polynomials tend to zero. We will thus assume that $\{q_n\}$ contains infinitely many integers which we list in order $0 \leqslant q_0 < q_1 < \ldots$. Since $\delta\{p_n\} = 1$ is assumed, $\delta\{q_n\} = 0$, or $q_n/n \to \infty$. Using duality, it will be possible to bring the Fabry gap theorem into play once again.

THEOREM 5. Let D be an open, connected, simply connected set in the plane with $0 \in D$. Let $f(z)$ be holomorphic on D, with

$$f(z) = \sum_{n=0}^{\infty} a_n z^{p_n}$$

near $z = 0$, where $\{p_n\}$ has density one. Then there exists a sequence of polynomials $\{P_k(z)\}$, containing only powers z^{p_n}, such that $\lim_{k\to\infty} P_k(z) = f(z)$, uniformly on every compact subset of D.

PROOF. As in Theorem 1, it will suffice to take an arbitrary compact set $K \subset D$ and show that $f(z)$ can be approximated as closely as desired on K by a polynomial $P(z)$ having only powers z^{p_n}.

Let γ be a rectifiable Jordan curve in D such that $\{0\} \cup K \subset D_\gamma$, where D_γ is the interior Jordan domain bounded by γ. Similarly, let Γ be another Jordan curve in D for which $\bar{D}_\gamma \subset D_\Gamma \subset D$. The maximum principle guarantees that $\|f-P\|_K \leqslant \|f-P\|_{\bar{D}_\gamma} = \|f-P\|_\gamma$ holds for any polynomial P. Thus it is sufficient to show that $f(z)$ can be approximated on γ by a polynomial $P(z)$ of the desired type.

In order to prove that $f(z)$ can be uniformly approximated on γ, let μ be any complex Borel measure on γ for which $\int_\gamma z^{p_n} d\mu(z) = 0$, $n = 0, 1, 2, \ldots$. We need only prove that $\int_\gamma f(z) d\mu(z) = 0$ follows. The equivalence of such a condition is an immediate consequence of the Hahn-Banach and Riesz representation theorems. For $z \in \gamma$ we have

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta,$$

and hence

$$\int_\gamma f(z) d\mu(z) = \frac{1}{2\pi i} \int_\gamma \left[\int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta \right] d\mu(z)$$

$$= \int_\Gamma f(\zeta) \left[\frac{1}{2\pi i} \int_\gamma \frac{d\mu(z)}{\zeta - z} \right] d\zeta = - \int_\Gamma f(\zeta) g_e(\zeta) d\zeta.$$

The "Cauchy transform"

$$g_e(\zeta) = \frac{1}{2\pi i} \int_\gamma \frac{d\mu(z)}{z - \zeta}$$

of the measure μ is holomorphic on the exterior domain bounded by γ. Moreover, for $|\zeta|$ sufficiently large, uniform convergence of $\sum_{k=0}^{\infty} (z/\zeta)^k$ on γ implies that

$$-2\pi i g_e(\zeta) = \frac{1}{\zeta} \int_\gamma \frac{d\mu(z)}{1 - z/\zeta} = \sum_{k=0}^{\infty} \frac{1}{\zeta^{k+1}} \int_\gamma z^k d\mu(z).$$

By the assumed orthogonality of the powers z^{p_n} to μ we have

$$g_e(\zeta) = \sum_{n=0}^{\infty} \frac{b_n}{\zeta^{q_n+1}} \tag{7}$$

for all sufficiently large $|\zeta|$.

The function $g_e(\zeta)$ has a removable singularity at ∞. Applying the Fabry gap theorem to $h(\zeta) = g_e(1/\zeta)$ we see that the series (7) must converge throughout the region $\{|z| > \varrho\}$, where $\varrho = d(0, \gamma)$. Indeed, the circle $\{|z| = \varrho\}$ could be the natural boundary for $g_e(\zeta)$, but no circle $\{|z| = \varrho'\}$, $\varrho' > \varrho$, could be the natural boundary since such a circle contains a point of holomorphy for $g_e(\zeta)$. It follows that the series (7) converges uniformly on Γ so that

$$\int_\gamma f(z)d\mu(z) = -\int_\Gamma f(\zeta)\sum_{n=0}^{\infty}\frac{b_n}{\zeta^{q_n+1}}d\zeta = -\sum_{n=0}^{\infty} b_n \int_\Gamma \frac{f(\zeta)}{\zeta^{q_n+1}}d\zeta. \tag{8}$$

However, for each n we have

$$\int_\Gamma \frac{f(\zeta)}{\zeta^{q_n+1}}d\zeta = \int_{|\zeta|=\varrho}\sum_{k=0}^{\infty} a_k\zeta^{p_k}\frac{d\zeta}{\zeta^{q_n+1}} = \sum_{k=0}^{\infty} a_k \int_{|\zeta|=\varrho}\zeta^{p_k-q_n-1}d\zeta$$

since the series for $f(\zeta)$ converges uniformly on $\{|\zeta|=\varrho\}$. Since $p_k \neq q_n$ for all k and n, each integral in the last series is zero. Thus from (8), $\int_\gamma f(z)d\mu(z)=0$ also.

§ 7. PRELIMINARIES FOR A WALSH RESULT, DENSITY ONE

In this section we prove a special case of Theorem 6 stated below. The proof begun here is further refined in § 8 where Theorem 6 is proven in complete generality with the aid of the local F. and M. Riesz theorem.

THEOREM 6. Let D be a Jordan domain with $0 \in D$ and let $\gamma = \partial D$ be of class C^1. Let $f(z)$ be holomorphic on D, continuous on $\bar{D}$, with

$$f(z) = \sum_{n=0}^{\infty} a_n z^{p_n}$$

near $z=0$. Assume finally that $\{p_n\}$ has density one. Then there exists a sequence of polynomials $\{P_k(z)\}$, containing only powers z^{p_n}, for which $\lim_{k\to\infty} P_k(z) = f(z)$, uniformly on $\bar{D}$.

The beginning of a proof. By the maximum principle it suffices to show that $f(z)$ can be uniformly approximated on γ by such polynomials. As in the proof of Theorem 5, this will be the case if and only if for every complex Borel measure μ on γ which satisfies

$$\int_\gamma z^{p_n}d\mu(z)=0,\ n=0, 1, 2, \ldots \tag{9}$$

we also have

$$\int_\gamma f(z)d\mu(z)=0. \tag{10}$$

It has been shown in Theorem 2 that when D is strongly starlike relative to $z=0$, the desired result is true without any restriction on the p_n. We will now use the Fabry gap theorem and a geometric construction to reduce the case of a general Jordan domain D to that for strongly starlike domains.

We may assume without loss of generality that $\{|z|<1\}$ is the largest open disc centered at the origin which is contained in D. Any point $\zeta \in \gamma = \partial D$ for which $|\zeta|=1$ will be called an inner extreme point of γ.

We wish to construct a simple closed curve $\gamma^* \subset \bar{D}$ whose interior domain D^* contains the origin and is strongly "starlike" relative to 0. Moreover, γ^* is to lie outside the disc $\{|z|<1\}$. Such a curve γ^* can be constructed from a finite number of "starlike" subarcs of γ (subarcs which no ray from the origin intersects more than once), each containing at least one inner extreme point, and a finite number of arcs Γ_ϱ of a circle $\{|z|=\varrho\}$ with $\varrho>1$. To see this, consider the subarcs γ_1 of γ which have both terminal points on $\{|z|=1\}$, but otherwise lie in $\{|z|>1\}$. Only finitely many of these arcs γ_1 can be non-starlike (at a limit point of initial points of non-starlike arcs γ_1, on $\{|z|=1\}$, the smoothness of γ would be violated). In forming γ^* out of γ, we retain the subarcs complementary to the non-starlike arcs γ_1, as well as small starlike subarcs of the latter adjacent to their terminal points. For the rest, each non-starlike arc γ_1 is replaced by an arc Γ_ϱ of the circle $\{|z|=\varrho\}$, with ϱ sufficiently close to 1. Fig. 2 contains an illustration of the desired construction.

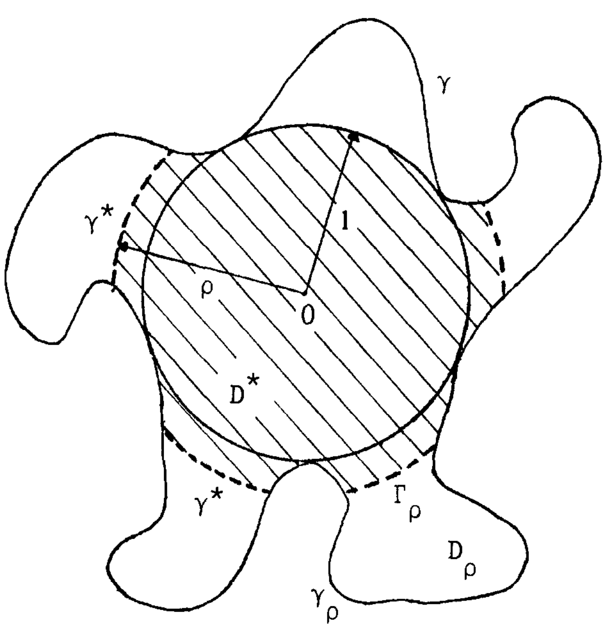

Fig. 2.

In the discussion to follow we will refer to the domains interior and exterior to γ as D and E respectively, while D^* and E^* will represent the domains interior and exterior to γ^*. Our proof will again make use of the Cauchy transform

$$\frac{1}{2\pi i}\int_\gamma \frac{d\mu(\zeta)}{\zeta-z} = \begin{cases} g_i(z), & z\in D \\ g_e(z), & z\in E \end{cases} \tag{11}$$

of the measure μ for which (9) is assumed to hold.

If μ has the special form $d\mu(z)=\varphi(z)dz$, where $\varphi(z)$ satisfies a Hölder-Lipschitz condition,

$$|\varphi(z)-\varphi(z')|\leqslant K|z-z'|^{\alpha} \quad (0<\alpha\leqslant 1) \tag{12}$$

for all z, $z'\in\gamma$, a relatively simple derivation of (10) can be given. It is known ([10], ch. 3) that for smooth curves γ condition (12) implies that both $g_i(z)$ and $g_e(z)$ have continuous extensions to the closures of their respective domains, and that the Plemelj formula

$$\varphi(z)=g_i(z)-g_e(z)$$

holds for all $z\in\gamma$. Thus with $f(z)$ and $g_i(z)$ holomorphic on D and continuous on $\bar{D}$, we have

$$\int_{\gamma} f(z)\varphi(z)dz=-\int_{\gamma} f(z)g_e(z)dz,$$

so that (10) follows once we show $f(z)$ is orthogonal to $g_e(z)$ on γ. Because of conditions (9), $g_e(z)$ has the expansion

$$g_e(z)=\sum_{n=0}^{\infty}\frac{b_n}{z^{q_n+1}} \tag{13}$$

for large $|z|$ (cf. § 6). Using the Fabry gap theorem we conclude (since $\delta\{q_n\}=0$) that (13) represents a holomorphic extension of $g_e(z)$ throughout $\{|z|>1\}$. Applying Cauchy's theorem to each of the Jordan domains D_ϱ which lie interior to γ but exterior to γ^* (see fig. 2) we have

$$\int_{\gamma} f(z)g_e(z)dz=\int_{\gamma^*} f(z)g_e(z)dz.$$

By Theorem 2 we may approximate $f(z)$ uniformly on γ^* by polynomials of the form $\Sigma b_n z^{p_n}$, so that using (13) we easily obtain $\int_{\gamma^*} f(z)g_e(z)dz=0$.

We remark finally that the measure μ could be assumed to have the form $d\mu(z)=\varphi(z)dz$ with $\varphi(z)$ satisfying (12), provided that f' and f'' (as well as f) have continuous extensions to $\bar{D}$. To prove that (10) holds for a general Borel measure μ on γ which satisfies (9) (without placing such additional restrictions on f) requires a more sophisticated argument.

§ 8. PROOF OF THEOREM 6 VIA THE LOCAL F. AND M. RIESZ THEOREM

Our proof of Theorem 6 will make use of the following local version of a classical result due to F. and M. Riesz.

THEOREM C ([10], p. 157–158). Let L be a rectifiable simple closed curve with interior domain D. Let $F(z)$ be holomorphic on D and have nontangential boundary values a.e. on L which define an integrable function $F(\zeta)$, and suppose that $F(z)$ is represented on D by the Cauchy integral

$$F(z)=\frac{1}{2\pi i}\int_{L}\frac{F(\zeta)d\zeta}{\zeta-z}.$$

Suppose furthermore that $F(\zeta)$ is of bounded variation (relative to arc length) on a closed subarc $l \subset L$. Then

(i) on each closed subarc of the open arc l°, $F(\zeta)$ is equal a.e. to an absolutely continuous function,

and

(ii) $F'(z)$ has nontangential limits at almost all points $\zeta \in l$ which are equal a.e. on l to $(d/d\zeta)F(\zeta)$ (derivative along l).

Reduction of the proof of Theorem 6. We need only show that (10) holds, where μ is an arbitrary complex Borel measure on γ whose moments (9) vanish. It is easily seen that we may assume $p_0=0$ and $p_1=1$, i.e. that

$$\int_\gamma d\mu(z) = \int_\gamma z\,d\mu(z) = 0. \tag{14}$$

Indeed, if $f(z)$ is a uniform limit of polynomials $P_k(z)$ containing nonzero linear parts $\alpha_k + \beta_k z$, while $f(0)=0$ and/or $f'(0)=0$, then the coefficients α_k and β_k will tend to zero.

Having (14) and using Theorem C, it will be possible to sweep the measure μ to a starlike curve $\gamma^* \subset \bar{D}$ of the form described in § 7. More precisely, *we will construct* γ^* *and a measure* μ^* *on* γ^* *such that*

$$\int_\gamma z^n d\mu(z) = \int_{\gamma^*} z^n d\mu^*(z), \quad n \geqslant 0. \tag{15}$$

By Walsh's theorem, there are polynomials in z which converge uniformly to $f(z)$ on $\bar{D}$, hence in particular on $\gamma \cup \gamma^*$. Thus (15) will imply that

$$\int_\gamma f(z)d\mu(z) = \int_{\gamma^*} f(z)d\mu^*(z). \tag{16}$$

On the other hand, conditions (9) and (15) show that $\int_{\gamma^*} z^{p_n} d\mu^*(z) = 0$, $n \geqslant 0$, so by the lacunary Walsh result for starlike domains (Theorem 2) we have $\int_{\gamma^*} f(z)d\mu^*(z)=0$. But then from (16) we will obtain $\int_\gamma f(z)d\mu(z)=0$, as desired.

Construction of the new measure μ^*. In connection with the determination of an appropriate starlike curve γ^* (cf. § 7), we introduce some additional notation. For each $\sigma \geqslant 1$, we let Γ_σ denote any of the closed subarcs of $\{|z|=\sigma\} \cap \bar{D}$ which have both terminal points on γ, but which otherwise belong to D. Similarly, γ_σ will denote any of the closed subarcs of $\gamma \cap \{|z| \geqslant \sigma\}$ whose terminal points lie on $\{|z|=\sigma\}$, but which otherwise lie in $\{|z|>\sigma\}$. Each pair of corresponding arcs γ_σ and Γ_σ bounds a Jordan domain D_σ (cf. fig. 2). As in § 7, the curve γ^* is constructed out of finitely many starlike arcs of γ and finitely many arcs Γ_ϱ of a circle $\{|z|=\varrho\}$. We will see later how the number $\varrho>1$ must be selected.

The measure μ^* on γ^* for which (15) is to be verified will have the form

$$d\mu^*(z)=\begin{cases}[g_i(z)-g_e(z)]dz & \text{on the arcs } \Gamma_\varrho\subset\gamma^*,\\ d\mu(z) & \text{on } \gamma\cap\gamma^*,\end{cases}\tag{17}$$

where g_i and g_e are the Cauchy transforms (11). As observed in § 7, $g_e(z)$ has an extension which is holomorphic throughout $\{|z|>1\}$ and hence also on each Γ_ϱ. The number ϱ will be chosen below so that $g_i(z)$ is integrable over the arcs Γ_ϱ. The postulated equality of μ and μ^* on $\gamma\cap\gamma^*$ means that (15) will be established if we show that

$$\int_{\cup\mathring{\gamma}_\varrho} z^n d\mu(z)=\int_{\cup\Gamma_\varrho} z^n[g_i(z)-g_e(z)]dz.\tag{18}$$

Let ζ_0 be any fixed point of γ and define for $\zeta\in\gamma$

$$m(\zeta)=\int_{[\zeta_0,\zeta)} d\mu(z),\quad M(\zeta)=\int_{[\zeta_0,\zeta)} m(z)dz,$$

where both integrals are taken in the positive direction along γ. Both functions are single-valued on γ (by (14)), $m(\zeta)$ is of bounded variation on γ and $M(\zeta)$ satisfies a Lipschitz condition on γ. The Cauchy transforms

$$h(z)=\frac{1}{2\pi i}\int_\gamma\frac{dm(\zeta)}{\zeta-z},\quad H(z)=\frac{1}{2\pi i}\int_\gamma\frac{dM(\zeta)}{\zeta-z}$$

when restricted to D will be denoted by $h_i(z)$ and $H_i(z)$, and when restricted to E, by $h_e(z)$ and $H_e(z)$. Clearly, $h'(z)=g(z)$ and $H'(z)=h(z)$ everywhere off γ. The Fabry gap theorem guarantees that h_e and H_e have holomorphic extensions (also called h_e and H_e) to $\{|z|>1\}$ just as g_e. Moreover, since $M(\zeta)$ satisfies a Lipschitz condition on γ, H_i and H_e have continuous extensions to $\bar{D}$ and $\bar{E}$ respectively, and the Plemelj formula

$$M(\zeta)=H_i(\zeta)-H_e(\zeta)\tag{19}$$

holds for all $\zeta\in\gamma$.

For any of the Jordan domains D_1 bounded by arcs Γ_1 and γ_1 we have

$$H_i(z)=\frac{1}{2\pi i}\int_{\partial D_1}\frac{H_i(\zeta)}{\zeta-z}d\zeta.$$

Using (19) and applying Theorem C, we conclude that $H_i(\zeta)$ is absolutely continuous on every closed subarc of $\mathring{\gamma}_1$, that $H_i'(z)=h_i(z)$ has non-tangential limits a.e. on γ_1, and that these limits coincide a.e. with $(d/d\zeta)H_i(\zeta)$ (derivative along γ). For $\zeta\in\mathring{\gamma}_1$ we will use $h_i(\zeta)$ to designate both the boundary values of $h_i(z)$ as well as $(d/d\zeta)H_i(\zeta)$.

We now consider numbers $\sigma > 1$ (but arbitrarily close to 1) so that the arc $\Gamma_\sigma = \{|z| = \sigma\} \cap \bar{D}_1$ meets γ_1 nontangentially and is such that limits of $h_i(z)$ along Γ_σ exist. We thus have $h_i(\zeta)$ integrable over $\partial D_\sigma = \Gamma_\sigma \cup \gamma_\sigma$, and (with proper orientation)

$$\int_{\partial D_\sigma} \zeta^n h_i(\zeta) d\zeta = \zeta^n H_i(\zeta)|_{\partial D_\sigma} - n \int_{\partial D_\sigma} \zeta^{n-1} H_i(\zeta) d\zeta = 0, \; n \geqslant 0.$$

The vanishing of the moments of $h_i(\zeta)$ on ∂D_σ implies that $h_i(z)$ belongs to the Smirnov class $E^1(D_\sigma)$ and that the Cauchy formula

$$h_i(z) = \frac{1}{2\pi i} \int_{\partial D_\sigma} \frac{h_i(\zeta)}{\zeta - z} d\zeta$$

holds for all $z \in D_\sigma$ ([10], p. 148–149). Differentiating the Plemelj formula (19) yields

(20) $$m(\zeta) = h_i(\zeta) - h_e(\zeta) \text{ a.e. on } \gamma_1.$$

Since $m(\zeta)$ is of bounded variation on γ and $h_e(z)$ is holomorphic on $\{|z| > 1\}$, $h_i(\zeta)$ is of bounded variation on $\gamma_\sigma \subset \partial D_\sigma$. Applying Theorem C again and remembering that σ could be arbitrarily close to 1, we conclude that $h_i(\zeta)$ is equal a.e. on the closed subarcs of $\mathring{\gamma}_1$ to an absolutely continuous function (also denoted by $h_i(\zeta)$), and that $h_i'(z) = g_i(z)$ has nontangential limits a.e. on γ_1 which coincide a.e. with $(d/d\zeta)h_i(\zeta)$. For $\zeta \in \mathring{\gamma}_1$ we may thus denote $(d/d\zeta)h_i(\zeta)$ by $g_i(\zeta)$.

We finally choose $\varrho > 1$ (but close to 1) so that each arc Γ_ϱ which, in the construction of γ^*, is to replace a non-starlike part γ_ϱ of a (non-starlike) subarc $\gamma_1 \subset \gamma$, satisfies the following conditions: Γ_ϱ meets γ_1 nontangentially and is such that $g_i(z)$ has finite limits along Γ_ϱ, and $h_i(z)$ is continuous on $\Gamma_\varrho \cup \gamma_\varrho$. From (20), $m(\zeta)$ is equal a.e. on the closed subarcs of $\mathring{\gamma}_1$ to an absolutely continuous function, and differentiation of (20) yields

$$d\mu(\zeta) = m'(\zeta) d\zeta = [h_i'(\zeta) - h_e'(\zeta)] d\zeta = [g_i(\zeta) - g_e(\zeta)] d\zeta,$$

each term considered as a measure on (any closed subarc of) $\mathring{\gamma}_1$. Thus

(21) $$\int_{\mathring{\gamma}_\varrho} \zeta^n d\mu(\zeta) = \int_{\gamma_\varrho} \zeta^n [g_i(\zeta) - g_e(\zeta)] d\zeta, \; n \geqslant 0.$$

The proof of (18) will be complete once we show that the integral on the right side of (21) is equal to

$$\int_{\Gamma_\varrho} \zeta^n [g_i(\zeta) - g_e(\zeta)] d\zeta.$$

Since $\gamma_\varrho \cup \Gamma_\varrho = \partial D_\varrho$ we must show that the moments of $g_i - g_e$ vanish on (the properly oriented) boundary ∂D_ϱ. This is clear for the holomorphic function g_e, while for g_i it follows from the vanishing of the moments of the (continuous) antiderivative h_i on ∂D_ϱ.

As a final remark we observe that with a little more work it is possible to prove that $h_i(z) \in E^1(D)$ rather than $h_i(z) \in E^1(D_\sigma)$ (with $\sigma > 1$) as was done above.

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